MATH2050B 1920 HW1

TA's solutions to selected problems

Q5. Show that there is no natural numbers strictly between n and $n+1$.

Solution. Let $P(j)$ be the statement that there is no natural number strictly between j and $j+1$.

For the case $n=0$, if k is a natural number $0<k$, then k is a successor of some natural number k'. Because every natural number is greater than or equal to 0, so $k=k'+1\geq 0+1=1$. Then $k\geq 1$, so we cannot have $k<1$. So $P(0)$ is true.

Now, assume that for $j=0,1,2,\dots,n$, $P(j)$ is true. If $n+1<k$, then k is a successor of some natural number k', so $n+1<k=k'+1$. This shows $n<k'$. By induction hypothesis $n+1\geq k'$, so $n+2\leq k'+1=k$. Hence we cannot have $k<n+2$. So the $P(n+1)$ is also true. By the Principle of Mathematical induction, there is no natural numbers strictly between n and $n+1$.

Q6. Given a set A of real numbers and a real number u, define "u is an upper bound of A" and its negation. Given the definition of "largest element of A". Give an example showing that an upper bound of A may not be an element of A and that a set A need not have the largest element.

Solution. u is said to be an upper bound of A if $u\geq a$ for every $a\in A$. u is not an upper bound of A if there is some $a\in A$ with $a>u$.

An element $a\in A$ is called largest if $a\geq x$ for all $x\in A$. If $A=\{x\in\mathbb{R}:x<1\}$, then 2 is an upper bound of A, but 2 is not an element in A. A in this case also does not admit any largest element, because if $a\in A$ is the largest, then there exists ϵ such that $a+\epsilon\in A$. This contradicts to the maximality of a.

Q8. Given a lower bound c for a non-empty set B of real numbers define "c is a greatest lower bound for B" and its negation.

Solution. c is said to be a greatest lower bound of B if $c+\epsilon$ is not a lower bound for B for all $\epsilon>0$.

Other definitions are possible, e.g. c is said to be a greatest lower bound of B if d is a lower bound of B, then $c\geq d$.

c is not a greatest lower bound for B means that there is a lower bound d for B with $c<d$.